

On the study of modified (p, q) - Bernstein polynomials and their applications



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ABSTRACT

The main aim of this article is to construct modified (p, q) - Bernstein polynomials which different from the (p, q) - Bernstein polynomials. We establish some new formulas and identities for Euler and Bernoulli polynomials and Stirling numbers of the second kind. Furthermore, we investigate some new properties by using these new polynomials arising from (p, q) -calculus. We also show our new polynomials and their generating function visually for some special values.

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1. Introduction

Many researchers in different fields are keenly interested in the study of Bernstein polynomials from the day which it was invited to today. These studies involve both pure and applied branches as mathematics, statistics, numerical analysis, CAGD.

The n th Bernstein polynomials, named after Bernstein (1912), are defined as

$$B_{k,n}(x) = \binom{n}{k} x^k (1-x)^{n-k},$$

where $\binom{n}{k} = \frac{n!}{k!(n-k)!}$ for $k \leq n$ and $x \in [0, 1]$.

After, about a century later, definition of the Bernstein polynomials, the generating function of these polynomials was obtained by Acikgoz and Araci (2010) as follows:

$$\sum_{n=0}^{\infty} B_{k,n}(x) \frac{t^n}{n!} = \frac{(xt)^k}{k!} e^{(1-x)t}, t \in \mathbb{C}.$$

Identifying the generating functions is of major importance in mathematics and its fields such as number theory, combinatorics and so on. By using the generating function, the many Bernstein polynomials' properties were obtained. Also, the generating function played important role between Bernstein polynomials and special numbers and polynomials and provided to arise new type definitions based on q -calculus (Acikgoz et al., 2010; 2012; Kim et al., 2010; Simsek, 2013; Simsek, 2017).

Let $0 < q < p \leq 1$. The (p, q) -analogue of n is defined by

$$[n]_{p,q} = \frac{p^n - q^n}{p - q}.$$

When $p = 1$ reduces to the q - analogue of n and also when $q \rightarrow p = 1$ reduces to the ordinary integers (Chakrabarti and Jagannathan, 1991; Duran, 2016; Sadjang, 2013). (p, q) - integers are generalization of q - integers such that we can write $[n]_{p,q}$ as below:

$$[n]_{p,q} = p^{n-1} [n]_{\frac{q}{p}}$$

If we take $p = 1$, we obtain q -integers but the opposite is not true. So, we can not derive $[n]_{p,q}$ by the aid of q - integers (Gupta, 2016).

Recently, the Bernstein polynomials have been moved into (p, q) - calculus and are studied by many researchers. Mursaleen et al. (2015a) described (p, q) -Bernstein polynomials. The (p, q) -Bernstein polynomials are defined as below:

$$B_{k,n}(x, p, q) = \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} p^{\binom{k}{2} - \binom{n}{2}} x^k (1-x)_{p,q}^{n-k}$$

where $0 < q < p \leq 1$ and $(1-x)_{p,q}^{n-k} = \prod_{j=0}^{n-k-1} (p^j - xq^j)$. In addition, they gave Bernstein-Schurer operators, Bernstein-Stancu operators, Bernstein Kantorovich operators depend on (p, q) - integers (Mursaleen et al., 2015a; 2015b; 2015c). Khan and Lobiyal (2015) obtained Lupaş Bernstein polynomials on (p, q) - integers and gave their some properties with related to parametric curves. Acar et al. (2016) proposed Kantorovich modifications of (p, q) - Bernstein operators and derived their rate of convergence, uniform convergence and approximations behaviors.

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We will explore in the next chapter that $\mathfrak{B}_{k,n}^{p,q}(x)$ polynomials have some properties such as generating function, recurrence relations, derivative property and an identity with related to two important elements of special polynomials and numbers.

2. Main results

In this part, we see that the generating function gives us useful properties about $\mathfrak{B}_{k,n}^{p,q}(x)$ polynomials. Furthermore, we investigate results of these polynomials under the (p, q) -calculus.

Definition 2.1: For $x \in [0,1]$, $k \leq n$ and $0 < q < p \leq 1$,

$$\mathfrak{B}_{k,n}^{p,q}(x) = \binom{n}{k} [x]_{p,q}^k [1-x]_{p,q}^{n-k}.$$

If we take $p = 1$, the $\mathfrak{B}_{k,n}^{p,q}(x)$ reduces to Kim's modified q - Bernstein polynomials. Also, as $q \rightarrow p = 1$, the $\mathfrak{B}_{k,n}^{p,q}(x)$ reduces to ordinary Bernstein polynomials. Some cases of the $\mathfrak{B}_{k,n}^{p,q}(x)$ polynomials are shown in Fig. 1.

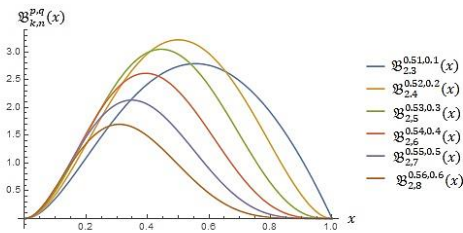


Fig. 1: The $\mathfrak{B}_{k,n}^{p,q}(x)$ polynomials for special values of p, q, k and n

Now, we show that $\mathfrak{B}_{k,n}^{p,q}(x)$ is described by linear combination of two $\mathfrak{B}_{k,n-1}^{p,q}(x)$ polynomials as below:

Corollary 2.2: For $x \in [0,1]$, $k \leq n$ and $0 < q < p \leq 1$, we have

$$\mathfrak{B}_{k,n}^{p,q}(x) = [1-x]_{p,q} \mathfrak{B}_{k,n-1}^{p,q}(x) + [x]_{p,q} \mathfrak{B}_{k-1,n-1}^{p,q}(x)$$

Proof: By using definition of $\mathfrak{B}_{k,n}^{p,q}(x)$ and property of Binomial coefficients, we obtain,

$$\begin{aligned} \mathfrak{B}_{k,n}^{p,q}(x) &= \binom{n}{k} [x]_{p,q}^k [1-x]_{p,q}^{n-k} = \binom{n-1}{k} + \\ &\binom{n-1}{k-1} [x]_{p,q} [1-x]_{p,q}^{n-k} = \binom{n-1}{k} [x]_{p,q} [1-x]_{p,q}^{n-k} + \\ &\binom{n-1}{k-1} [x]_{p,q}^2 [1-x]_{p,q}^{n-k} = [1-x]_{p,q} \binom{n-1}{k} [x]_{p,q}^k [1-x]_{p,q}^{n-k-1} \\ &+ [x]_{p,q} \binom{n-1}{k-1} [x]_{p,q}^{k-1} [1-x]_{p,q}^{n-k} = [1-x]_{p,q} \mathfrak{B}_{k,n-1}^{p,q}(x) + [x]_{p,q} \mathfrak{B}_{k-1,n-1}^{p,q}(x). \end{aligned}$$

We show that $\mathfrak{B}_{k,n}^{p,q}(x)$ polynomials have symmetric property as below:

Corollary 2.3: For $x \in [0,1]$, $k \leq n$ and $0 < q < p \leq 1$, we have

$$\mathfrak{B}_{n-k,n}^{p,q}(1-x) = \mathfrak{B}_{k,n}^{p,q}(x)$$

Proof: By substituting $x \rightarrow 1-x$ and $k \rightarrow n-k$ into above equation, we get

$$\begin{aligned} \mathfrak{B}_{n-k,n}^{p,q}(1-x) &= \binom{n}{n-k} [1-x]_{p,q}^{n-k} [1-1+x]_{p,q}^k = \\ &\binom{n}{k} [x]_{p,q}^k [1-x]_{p,q}^{n-k} = \mathfrak{B}_{k,n}^{p,q}(x). \end{aligned}$$

We see that $n-k$ th $\mathfrak{B}_{k,n-k}^{p,q}(x)$ polynomials are obtained by the aid of two $n+1$ th $\mathfrak{B}_{k,n+1}^{p,q}(x)$ polynomials by means of at the following corollary:

Corollary 2.4: For $x \in [0,1]$, $k \leq n$ and $0 < q < p \leq 1$, we have

$$\begin{aligned} (p^{-x} + (1-p^{-x}q^{1-x})[x]_{p,q}) \mathfrak{B}_{n-k,n}^{p,q}(x) = \\ \left(\frac{n-k+1}{k}\right) \mathfrak{B}_{k,n+1}^{p,q}(x) + \left(\frac{k+1}{n+1}\right) \mathfrak{B}_{k+1,n+1}^{p,q}(x) \end{aligned}$$

Proof: Proof of this corollary follows from definition of $\mathfrak{B}_{k,n}^{p,q}(x)$.

The expression of below corollary is to go from the $k-1$ modified polynomials to k modified polynomials n th degree.

Corollary 2.5: For $x \in [0,1]$, $k \leq n$ and $0 < q < p \leq 1$, we have

$$\mathfrak{B}_{k,n}^{p,q}(x) = \left(\frac{n-k+1}{k}\right) \left(\frac{[x]_{p,q}}{[1-x]_{p,q}}\right) \mathfrak{B}_{k-1,n}^{p,q}(x)$$

Proof: By applying the definition of $\mathfrak{B}_{k,n}^{p,q}(x)$ polynomials, we have

$$\begin{aligned} \mathfrak{B}_{k,n}^{p,q}(x) &= \binom{n}{k} [x]_{p,q}^k [1-x]_{p,q}^{n-k} = \left(\frac{n-k+1}{k}\right) \binom{n}{k-1} [x]_{p,q}^k [1-x]_{p,q}^{n-k} \\ &= \left(\frac{n-k+1}{k}\right) \left(\frac{[x]_{p,q}}{[1-x]_{p,q}}\right) [x]_{p,q}^{k-1} [1-x]_{p,q}^{n-k+1} = \\ &\left(\frac{n-k+1}{k}\right) \left(\frac{[x]_{p,q}}{[1-x]_{p,q}}\right) \mathfrak{B}_{k-1,n}^{p,q}(x). \end{aligned}$$

The desired result is shown.

In the next corollary, we give the identity by the aid of summation of (p, q) - integers and binomial expansion, respectively.

Corollary 2.6: For $x \in [0,1]$, $k \leq n$ and $0 < q < p \leq 1$, we have

$$\mathfrak{B}_{k,n}^{p,q}(x) = \sum_{l=0}^{n-k} \binom{k+l}{k} \binom{n}{k+l} p^{-(n+1-k)x} q^{(1-x)l} (-1)^l [x]_{p,q}^{k+l}.$$

Proof: By using of definition of $\mathfrak{B}_{k,n}^{p,q}(x)$ and binomial formula, we obtain

$$\begin{aligned} \mathfrak{B}_{k,n}^{p,q}(x) &= \binom{n}{k} [x]_{p,q}^k [1-x]_{p,q}^{n-k} = \binom{n}{k} [x]_{p,q}^k (p^{-x} + \\ &(1-p^{-x}q^{1-x})[x]_{p,q})^{n-k} = \\ &\binom{n}{k} [x]_{p,q}^k \sum_{l=0}^{n-k} \binom{n-k}{l} p^{-(n-k)x} p^{-xl} q^{(1-x)l} (-1)^l [x]_{p,q}^l = \\ &\sum_{l=0}^{n-k} \binom{k+l}{k} \binom{n}{k+l} p^{-(n+1-k)x} q^{(1-x)l} (-1)^l [x]_{p,q}^{k+l}. \end{aligned}$$

Therefore, we arrive at the desired result.

We give the reflection of $\mathfrak{B}_{k,n}^{p,q}(x)$ polynomials under derivative operator at the following corollary:

Corollary 2.7: For $x \in [0,1]$, $k \leq n$ and $0 < q < p \leq 1$, we have

$$\frac{d(\mathfrak{B}_{k,n}^{p,q}(x))}{dx} = \frac{\mathfrak{B}_{k-1,n}^{p,q}(x) \left[\frac{[1-x]_{p,q} (q^x \log q - p^x \log p)}{n-k} + \frac{[x]_{p,q} (q^{1-x} \log q - p^{1-x} \log p)}{k} \right]}{k}$$

Proof: Using the definition of derivative, it is seen that

$$\frac{d(\mathfrak{B}_{k,n}^{p,q}(x))}{dx} = \binom{n}{k} \left(\frac{1}{q-p}\right)^n \times [k(q^x - p^x)^{k-1} (q^x \log p - p^x \log q) (q^{1-x} - p^{1-x})^{n-k} + (n-k)(q^x - p^x)^k (p^{1-x} \log p - q^{1-x} \log q) (q^{1-x} - p^{1-x})^{n-k-1}]$$

and after making some algebraic operations, we obtain

$$\begin{aligned} \frac{d(\mathfrak{B}_{k,n}^{p,q}(x))}{dx} &= \binom{n}{k} \left(\frac{1}{q-p}\right)^n \times (q^x - p^x)^{k-1} (q^{1-x} - p^{1-x})^{n-k-1} \times \\ &\left[k(q^x \log p - p^x \log q) (q^{1-x} - p^{1-x}) + (n-k)(q^x - p^x) (p^{1-x} \log p - q^{1-x} \log q) \right] = \\ &\binom{n}{k} \left(\frac{1}{q-p}\right)^{n-2} [x]_{p,q}^{k-1} [1-x]_{p,q}^{n-k-1} \times (q^x - p^x)^{k-1} (q^{1-x} - p^{1-x})^{n-k-1} \times \\ &\left[k(q^x \log p - p^x \log q) [1-x]_{p,q} + (n-k)(q^x - p^x) (p^{1-x} \log p - q^{1-x} \log q) [x]_{p,q} \right] = \\ &\mathfrak{B}_{k-1,n}^{p,q}(x) \left[\frac{(q^x \log p - p^x \log q) [1-x]_{p,q}}{k} + \frac{(p^{1-x} \log p - q^{1-x} \log q) [x]_{p,q}}{k} \right]. \end{aligned}$$

Therefore, the proof of corollary is completed.

3. The generating function of modify (p, q)-Bernstein polynomials

Now, we give the generating function of our modified polynomials at the following theorem:

Theorem 3.1: For $x \in [0,1]$, $k \leq n$ and $0 < q < p \leq 1$, we have

$$F_k^{p,q}(t; x) = \sum_{n=0}^{\infty} \mathfrak{B}_{k,n}^{p,q}(x) \frac{t^n}{n!} = \frac{[x]_{p,q}^k t^k}{k!} e^{[1-x]_{p,q} t},$$

where $F_k^{p,q}(t; x)$ is called generating function of $\mathfrak{B}_{k,n}^{p,q}(x)$ polynomials.

Proof: The proof follows from the $\mathfrak{B}_{k,n}^{p,q}(x)$ polynomials that

$$\begin{aligned} F_k^{p,q}(t; x) &= \sum_{n=0}^{\infty} \mathfrak{B}_{k,n}^{p,q}(x) \frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty} \binom{n}{k} [x]_{p,q}^k [1-x]_{p,q}^{n-k} \frac{t^n}{n!} \\ &= \frac{[x]_{p,q}^k t^k}{k!} \sum_{n=0}^{\infty} [1-x]_{p,q}^{n-k} \frac{t^{n-k}}{(n-k)!} \\ &= \frac{[x]_{p,q}^k t^k}{k!} \sum_{n=k}^{\infty} [1-x]_{p,q}^n \frac{t^n}{n!} = \frac{[x]_{p,q}^k t^k}{k!} e^{[1-x]_{p,q} t}. \end{aligned}$$

This ends the proof.

The generating function is shown for special values of p, q and k in Fig. 2.

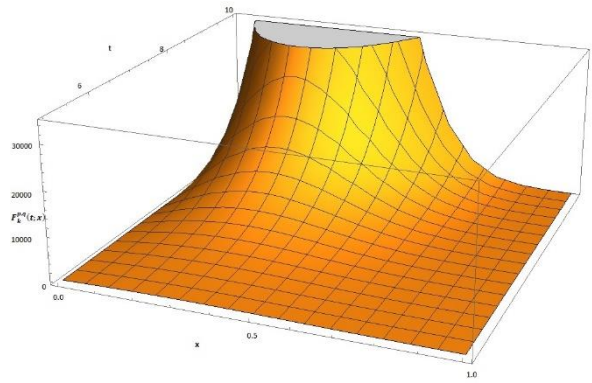


Fig. 2: The generating function of $\mathfrak{B}_{k,n}^{p,q}(x)$ polynomials for special values of p, q, k and n

The modified (p, q) - Bernstein polynomials have the following properties with related to the generating function.

Corollary 3.2: For $x \in [0,1]$, $k \leq n$ and $0 < q < p \leq 1$, we have

$$\frac{\partial^v (F_k^{p,q}(t; x))}{\partial t^v} = \sum_{j=0}^v \mathfrak{B}_{j,v}^{p,q}(x) F_{k-j}^{p,q}(t; x).$$

Proof: To show proof of this corollary, let $F_k^{p,q}(t; x)$ be as follow:

$$A = \frac{[x]_{p,q}^k t^k}{k!}, B = e^{[1-x]_{p,q} t}.$$

By applying the Leibniz rule to $F_k^{p,q}(t; x)$ depend on t , we have

$$\begin{aligned} \frac{\partial^v (F_k^{p,q}(t; x))}{\partial t^v} &= \sum_{j=0}^v \binom{v}{j} \frac{[x]_{p,q}^k t^{k-j}}{(k-j)!} [1-x]_{p,q}^{v-j} e^{[1-x]_{p,q} t} \\ &= \sum_{j=0}^v \binom{v}{j} [x]_{p,q}^j [1-x]_{p,q}^{v-j} \frac{[x]_{p,q}^{k-j} t^{k-j}}{(k-j)!} e^{[1-x]_{p,q} t} \\ &= \sum_{j=0}^v \mathfrak{B}_{j,v}^{p,q}(x) F_{k-j}^{p,q}(t; x). \end{aligned}$$

Therefore, we obtain the desired result.

Corollary 3.3: For $x \in [0,1]$, $k \leq n$ and $0 < q < p \leq 1$, we have

$$\begin{aligned} &(p^{-x} + (1 - p^{-x} q^{1-x}) [x]_{p,q}) \mathfrak{B}_{k,n}^{p,q}(x) \\ &= \frac{1}{n+1} \left((k+1) \mathfrak{B}_{k+1,n+1}^{p,q}(x) \right) \\ &+ (n+1-k) \mathfrak{B}_{k,n+1}^{p,q}(x). \end{aligned}$$

Proof: Firstly, we consider the following identity from the generating function of $\mathfrak{B}_{k,n}^{p,q}(x)$,

$$([x]_{p,q} t)^b F_k^{p,q}(t; x) = \frac{(k+b)!}{k!} F_k^{p,q}(t; x).$$

From above equation, we obtain

$$[x]_{p,q}^b \mathfrak{B}_{k,n}^{p,q}(x) = \frac{n!(k+b)!}{k!(n+b)!} \mathfrak{B}_{k+b,n+b}^{p,q}(x).$$

If we take $b = 1$ at the above equation,

$$[x]_{p,q} \mathfrak{B}_{k,n}^{p,q}(x) = \frac{k+1}{n+1} \mathfrak{B}_{k+1,n+1}^{p,q}(x). \tag{1}$$

On the other hand, the generating function can be rewritten in the following form:

$$([x]_{p,q}t)^{-b} F_k^{p,q}(t; x) = \frac{(k-b)!}{k!} F_{k-b}^{p,q}(t; x).$$

By aid of the definition of generating function, we have

$$[1-x]_{p,q}^b \mathfrak{B}_{k,n}^{p,q}(x) = \frac{n!(n+b-k)!}{(n-k)!(n+b)!} \mathfrak{B}_{k,n+b}^{p,q}(x).$$

By choosing $b = 1$ at the above equation, we get

$$[1-x]_{p,q} \mathfrak{B}_{k,n}^{p,q}(x) = \frac{(n+1-k)}{(n+1)} \mathfrak{B}_{k,n+1}^{p,q}(x). \tag{2}$$

Combining Eq. (1) and Eq. (2), we arrive the desired result.

Corollary 3.4: For $x \in [0,1]$, $k \leq n$ and $0 < q < p \leq 1$, we have

$$\sum_{j=0}^{n-k} \binom{n}{j} \mathfrak{B}_{k,n-j}^{p,q}(x) (p^{-x}q^{1-x})^j [x]_{p,q}^{j-k} = p^{-x(n-k)} \binom{n}{k}$$

Proof: From the definition of generating function, we observe that;

$$\sum_{n=0}^{\infty} \mathfrak{B}_{k,n}^{p,q}(x) \frac{t^n}{n!} e^{(p^{-x}q^{1-x}[x]_{p,q})t} = \frac{[x]_{p,q}^k t^k}{k!} e^{p^{-x}t}.$$

On the other hand, it is easy to see that

$$\sum_{n=0}^{\infty} \mathfrak{B}_{k,n}^{p,q}(x) \frac{t^n}{n!} \sum_{n=0}^{\infty} (p^{-x}q^{1-x})^n \frac{[x]_{p,q}^n}{n!} = \frac{[x]_{p,q}^k t^k}{k!} \sum_{n=0}^{\infty} \frac{(p^{-x}t)^n}{n!}.$$

Then we apply the Cauchy product and compare coefficients of $\frac{t^n}{n!}$ as follows:

$$\sum_{n=k}^{\infty} \left(\sum_{j=0}^{n-k} \binom{n}{j} \mathfrak{B}_{k,n-j}^{p,q}(x) (p^{-x}q^{1-x})^j [x]_{p,q}^j \right) \frac{t^n}{n!} = [x]_{p,q}^k \sum_{n=k}^{\infty} p^{-x(n-k)} \binom{n}{k} \frac{t^n}{n!}.$$

Thus, the proof is completed.

Theorem 3.5: For $x \in [0,1]$, $k \leq n$ and $0 < q < p \leq 1$, we have

$$\sum_{l=0}^{n-k} \binom{n}{l} (-p^{-x})^l \mathfrak{B}_{k,n-l}^{p,q}(x) = \binom{n}{k} (-1)^{n-k} (p^{-x}q^{1-x})^{n-k} [x]_{p,q}^n.$$

Proof: If we arrange the generating function, we have

$$\left(\sum_{n=0}^{\infty} \mathfrak{B}_{k,n}^{p,q}(x) \frac{t^n}{n!} \right) e^{-p^{-x}t} = \frac{[x]_{p,q}^k t^k}{k!} e^{(-p^{-x}q^{1-x}[x]_{p,q})t}.$$

After the Taylor series generated for $e^{-p^{-x}t}$, we get

$$\left(\sum_{n=0}^{\infty} \mathfrak{B}_{k,n}^{p,q}(x) \frac{t^n}{n!} \right) \left(\sum_{n=0}^{\infty} (-1)^n (p^{-x})^n \frac{t^n}{n!} \right) = \frac{[x]_{p,q}^k t^k}{k!} \left(\sum_{n=0}^{\infty} (-1)^n (p^{-x}q^{1-x}[x]_{p,q})^n \frac{t^n}{n!} \right).$$

By applying the Cauchy product both sides of above equality, we obtain

$$\sum_{n=k}^{\infty} \left(\sum_{l=0}^{n-k} \binom{n}{l} \mathfrak{B}_{k,n-l}^{p,q}(x) (-p^{-x})^l \right) \frac{t^n}{n!} = \frac{[x]_{p,q}^k t^k}{k!} \sum_{n=0}^{\infty} (-1)^n (p^{-x}q^{1-x}[x]_{p,q})^n \frac{t^n}{n!}.$$

By comparison of coefficients of $\frac{t^n}{n!}$, the proof is completed.

4. Relations between modify (p,q)-Bernstein polynomials and special polynomials

In this section, we obtain some equalities interested in the modify polynomials the Euler polynomials, the Bernoulli polynomials and the Stirling numbers of the second kind.

Theorem 4.1:

$$\sum_{k=0}^n \binom{n}{k} \mathfrak{B}_{k,n}^{p,q}(x) E_{n-k} = \frac{[x]_{p,q}^k}{k!} G_n^{(k)}([1-x]_{p,q}).$$

Proof: By using the generating function of $\mathfrak{B}_{k,n}^{p,q}(x)$, we have

$$\begin{aligned} \sum_{n=0}^{\infty} \mathfrak{B}_{k,n}^{p,q}(x) \frac{t^n}{n!} &= \frac{[x]_{p,q}^k t^k}{k!} e^{[1-x]_{p,q}t} \\ &= \frac{[x]_{p,q}^k t^k}{k!} e^{[1-x]_{p,q}t} \frac{(e^t+1)^k 2^k}{(e^t+1)^k 2^k} \\ &= \left(\frac{2t}{e^t+1} \right)^k e^{[1-x]_{p,q}t} \frac{[x]_{p,q}^k}{k! 2^k} (e^t+1)^k \\ &= \frac{\left(\frac{2t}{e^t+1} \right)^k e^{[1-x]_{p,q}t} [x]_{p,q}^k}{\frac{2^k}{(e^t+1)^k} k!} \\ &= \frac{\sum_{n=0}^{\infty} G_n^{(k)}([1-x]_{p,q}) \frac{t^n}{n!} [x]_{p,q}^k}{\sum_{n=0}^{\infty} E_n \frac{t^n}{n!}}, \end{aligned}$$

where the generating functions of the Genocchi polynomials and the Euler numbers are defined as follows:

$$\frac{2t}{e^t+1} e^{xt} = \sum_{n=0}^{\infty} G_n(x) \frac{t^n}{n!}, |t| < \pi$$

and

$$\frac{2}{e^t+1} = \sum_{n=0}^{\infty} E_n \frac{t^n}{n!}, |t| < \pi.$$

If we arrange the above equality, we get

$$\left(\sum_{n=0}^{\infty} \mathfrak{B}_{k,n}^{p,q}(x) \frac{t^n}{n!} \right) \left(\sum_{n=0}^{\infty} E_n \frac{t^n}{n!} \right) = \sum_{n=0}^{\infty} G_n^{(k)}([1-x]_{p,q}) \frac{t^n [x]_{p,q}^k}{n! k!}.$$

Thereby, by applying the Cauchy product for above equality and then comparison the coefficients of $\frac{t^n}{n!}$, we complete the proof.

Theorem 4.2:

$$\mathfrak{B}_{k,n}^{p,q}(x) = [x]_{p,q} \sum_{l=0}^n \binom{n}{l} B_1^{(k)}([1-x]_{p,q}) S(n-k, l)$$

Proof: By aid of the $\mathfrak{B}_{k,n}^{p,q}(x)$ polynomials, we get

$$\begin{aligned} \sum_{n=0}^{\infty} \mathfrak{B}_{k,n}^{p,q}(x) \frac{t^n}{n!} &= \frac{[x]_{p,q}^k t^k}{k!} e^{[1-x]_{p,q}t} \\ &= \frac{[x]_{p,q}^k t^k}{k!} e^{[1-x]_{p,q}t} \frac{(e^t - 1)^k}{(e^t - 1)^k} \\ &= \left(\frac{t}{e^t - 1}\right)^k e^{[1-x]_{p,q}t} \frac{(e^t - 1)^k}{k!} [x]_{p,q}^k \\ &= \sum_{n=0}^{\infty} B_n^{(k)}([1-x]_{p,q}) \frac{t^n}{n!} \sum_{n=0}^{\infty} S(n, k) \frac{t^n}{n!}, \end{aligned}$$

where the generating function of the Bernoulli polynomials and the second kind Stirling numbers are defined at the following equalities:

$$\frac{t}{e^t - 1} e^{xt} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!}, |t| < 2\pi$$

and

$$\frac{(e^t + 1)^k}{k!} = \sum_{n=0}^{\infty} S(n; k) \frac{t^n}{n!}.$$

By applying the Cauchy product at above equation as in the previous corollary, the desired result is obtained.

5. Conclusion

In this study, we extended the Bernstein polynomials to new type (p,q)-Bernstein polynomials. We also obtained some useful properties such as the generating function, symmetric function, recurrence relations and equalities under the derivative operators for these polynomials. We plotted the graphs of these polynomials and their generating function. Furthermore, some of our results are generalizations results in Kim et al. (2010) and Simsek (2017). Because of the Bernstein polynomials are very useful in many different fields such as statistics, engineering and CAGD, these new polynomials can be used in the mentioned areas.

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